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Report Title

Final Report

ABSTRACT

Worked performed during this period includes the investigation into eigenvalue behavior of several different classes of large dimensional random matrices. They are: 1) a class of random matrices important to array signal processing and wireless communications with the goal of proving exact separation of their eigenvalues; 2) an ensemble of random matrices used to estimate the powers transmitted by multiple signal sources in multi-antenna fading channels; 3) another ensemble whose eigenvalues yield the mutual information of a multiple antenna radio channel, for which a central limit theorem is proven; 4) ensembles which yield robust estimation of a population covariance matrix with application to array signal processing; and 5) a sample covariance matrix for which a CLT is studied on linear statistics of its eigenvalues, whose limiting empirical distribution of its eigenvalues is studied with application toward computing the power of a likelihood ratio test for determining the presence of spike eigenvalues in the population covariance matrix.

Enter List of papers submitted or published that acknowledge ARO support from the start of the project to the date of this printing. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

Received		Paper
08/30/2012	6.00	Romain Couillet, Merouane Debbah, Jack W. Silverstein. A Deterministic Equivalent for the Capacity Analysis of Correlated Multi-User MIMOChannels, IEEE TRANSACTIONS ON INFORMATION THEORY, (06 2011): 3493. doi:
08/30/2012	7.00	Zhidong Bai, Jack W. Silverstein. No Eigenvalues Outside the Support of the LimitingSpectral Distribution of Information-Plus-NoiseType Matrices, Random Matrices: Theory and Applications , (01 2012): 1150004. doi:
08/30/2012	8.00	Walid Hachem, Malika Kharouf, Jamal Najim, Jack W. Silverstein. A CLT FOR INFORMATION-THEORETIC STATISTICS OFNON-CENTERED GRAM RANDOM MATRICES, Random Matrices: Theory and Applications , (04 2012): 1150010. doi:

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(b) Papers published in non-peer-reviewed journals (N/A for none)

Received		Paper
08/30/2012	5.00	Merouane Debbah, Romain Couillet, Jack W. Silverstein, Zhidong Bai. Eigen-Inference for Energy Estimation of Multiple Sources, IEEE TRANSACTIONS ON INFORMATION THEORY, (04 2011): 2420. doi:

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Number of Papers published in non peer-reviewed journals:

(c) Presentations

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Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

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(d) Manuscripts

Received		Paper
02/24/2011	3.00	Zhidong Bai, Jack W. Silverstein. No Eigenvalues Outside the Support of the Limiting Spectral Distribution of Information-Plus-Noise Type Matrices, (02 2011)
04/27/2010	2.00	Romain Couillet, Jack W. Silverstein, Merouane Debbah. Eigen-Inference for Energy Estimation of Multiple Sources, (04 2010)
04/27/2010	1.00	Romain Couillet, Jack W. Silverstein, Merouane Debbah. `Eigen-inference for Multi-source Power Estimation, (04 2010)
08/10/2011	4.00	Malika Kharouf, Jamal Najim, Jack W. Silverstein, Walid Hachem. A CLT FOR INFORMATION-THEORETIC STATISTICS OFNON-CENTERED GRAM RANDOM MATRICES, Random Matrices: Theory and Applications (submitted) (06 2011)
08/30/2012	9.00	Romain Couillet, Frederic Pascal, Jack W. Silverstein. Robust M-Estimation for Array Processing: A Random Matrix Approach, IEEE Transactions on Signal Processing (04 2012)
12/14/2013	10.00	Romain Couillet , Frederic Pascal, Jack W. Silverstein. Robust Estimates of Covariance Matrices in the Large Dimensional Regime and Application to Array Processing, IEEE TRANSACTIONS ON Information Theory (10 2013)
12/14/2013	11.00	Qinwen Wang, Jack W. Silverstein, Jianfeng Yao. A Note on the CLT of the LSS for Sample Covariance Matrix from a Spiked Population Model , Journal of Multivariate Analysis (07 2013)
12/14/2013	12.00	Romain Couillet, Frederic Pascal, Jack W. Silverstein. The Random Matrix Regime of Maronna's M-estimatorwith elliptically distributed samples, Journal of Multivariate Analysis (11 2013)

TOTAL: **8**

Number of Manuscripts:

Books

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Patents Submitted

Patents Awarded

Awards

Graduate Students

NAME	PERCENT_SUPPORTED

FTE Equivalent:
Total Number:

Names of Post Doctorates

NAME	PERCENT_SUPPORTED

FTE Equivalent:
Total Number:

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NAME	PERCENT_SUPPORTED

FTE Equivalent:
Total Number:

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FTE Equivalent:
Total Number:

Student Metrics

This section only applies to graduating undergraduates supported by this agreement in this reporting period

The number of undergraduates funded by this agreement who graduated during this period: 0.00

The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields:...... 0.00

The number of undergraduates funded by your agreement who graduated during this period and will continue to pursue a graduate or Ph.D. degree in science, mathematics, engineering, or technology fields:...... 0.00

Number of graduating undergraduates who achieved a 3.5 GPA to 4.0 (4.0 max scale):...... 0.00

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Names of Personnel receiving masters degrees

NAME

Total Number:

Names of personnel receiving PHDs

NAME

Total Number:

Names of other research staff

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Sub Contractors (DD882)

Inventions (DD882)

Scientific Progress

See Attachment

Technology Transfer

Worked performed during this period includes the investigation into eigenvalue behavior of several different classes of large dimensional random matrices. They are: 1) a class of random matrices important to array signal processing and wireless communications with the goal of proving exact separation of their eigenvalues; 2) an ensemble of random matrices used to estimate the powers transmitted by multiple signal sources in multi-antenna fading channels; 3) another ensemble whose eigenvalues yield the mutual information of a multiple antenna radio channel, for which a central limit theorem is proven; 4) ensembles which yield robust estimation of a population covariance matrix with application to array signal processing; and 5) a sample covariance matrix for which a CLT is studied on linear statistics of its eigenvalues, whose limiting empirical distribution of its eigenvalues is studied with application toward computing the power of a likelihood ratio test for determining the presence of spike eigenvalues in the population covariance matrix. Details are given below.

Information-plus-noise model. This matrix is defined as

$$C_n = (1/N)(R_n + \sigma X_n)(R_n + \sigma X_n)^*,$$

where, R_n and X_n are both $n \times N$, the entries of X_n are independent standardized random variables, and $\sigma > 0$. Here, both n and N are considered large and on the same order of magnitude. As mentioned in the previous interim reports the eigenvalues of C_n are significant to both array signal processing, where it models the sample correlation matrix of N "snapshots" or recordings taken from a bank of n antennas of signals (the columns of R_n) in a noise-filled environment (columns of σX_n), and in wireless communications, modeling schemes where the fading channel, $(1/\sqrt{N})(R_n + \sigma X_n)$, has non-zero means. Assume H_n, the empirical distribution function of the eigenvalues of $(1/N)R_n R_n^*$, ($H_n(x) = (1/n)$(number of eigenvalues of $(1/N)R_n R_n^* \leq x$)) converges in distribution to H as $n \to \infty$, and $\lim_{n\to\infty} n/N = c > 0$. Then it has been shown in reference [6] of the proposal that F_n, the empirical distribution function of the eigenvalues of C_n, converges in distribution, with probability one, to a non-random F, whose Stieltjes transform

$$m(z) = \int \frac{1}{\lambda - z} dF(\lambda) \qquad \Im z > 0$$

is the unique solution for $\Im z > 0$ to

$$(*) \qquad m = \int \frac{1}{\frac{t}{1+\sigma^2 cm} - (1 + \sigma^2 cm)z + \sigma^2(1-c)} dH(t).$$

Under additional assumptions on the entries of X_n the following has been shown in [1]. Assume $[a_1, a_2] \subset (b_1, b_2)$ where $b_1 > 0$ and (b_1, b_2) is outside the support of F. Let $c_n = n/N$ and F^{c_n, H_n} denote the distribution having Stieltjes transform which satisfies $(*)$ with c, H replaced by c_n, H_n respectively. A natural assumption is to impose (b_1, b_2) is also outside the support of F^{c_n, H_n} for all n large. Then, it is proven that

$$\mathsf{P}(\text{no eigenvalue of } C_n \text{ appears in } [a_1, b_1] \text{ for all } n \text{ large}) = 1.$$

Time was devoted on showing the second part of exact separation, namely the correct number of eigenvalues on either side of $[a_1, b_1]$. For C_n this should correspond to the number of eigenvalues of $(1/N)R_nR_n^*$ on either side of some interval (d_1, d_2). The strategy is similar to that of reference [2] of the proposal, which establishes the exact separation of the eigenvalues of a class of large dimensional sample covariance matrices. The idea is to systematically increase the number of columns of $(R_n + \sigma X_n)$, until the matrix $C_n' = (1/N')(R_n' + \sigma X_n')(R_n' + \sigma X_n')^*$ (N' being the number of columns of $R_n' + \sigma X_n'$) is close to $(1/N)R_n{R_n}^* + \sigma^2 I$, I being the $n \times n$ identity matrix. This can be achieved from known results on the extreme eigenvalues of $(1/N')X_n'{X_n'}^*$ and on results established by the principal investigator on the cross terms $(1/N')(R_n'{X_n'}^* + X_n'{R_n'}^*)$. The increase in the number of columns is done in a finite number of steps, where at each step no eigenvalue of C_n' will cross the interval $[a_1, b_1]$.

The machinery in performing the steps is all in place, except for one important factor. It is crucial that there is an interval in $[a_1, b_1]$ which does not shrink to a point as the number of columns increase, and the principle investigator has not yet shown this to be true. In reference [7] of the proposal, for each c_n, H_n, a function $x_n(b)$ for $b > 0$ is created which is associated with $[a_1, b_1]$ and increases on an interval, with minimum and maximum values being the endpoints of the interval which contains $[a_1, b_1]$ and is outside the support of F^{c_n, H_n}. From simulations this interval appears to be increasing in length as c_n decreases (corresponding to the addition of more columns), but the intervals are not simply nested, that is, not necessarily one outside the other. It is when the principal investigator shows the intervals remain larger than some positive length will he be able to complete the proof of exact separation. When he is able to do so, he will acknowledge funding from this grant.

Multi-source power estimation. Research was conducted on a way to estimate the transmit powers of several sources in mult-antenna fading channels, when the number of sensors and the number of samples are sufficiently large in comparison to the number of sources. Consider K stations which transmit data, where, for $k = 1, \ldots, K$, transmitter k has n_k antennas with (unknown) transmission power P_k. These stations transmit to a receiver, a collection of N sensing devices. Let H_k, $N \times n_k$ denote the channel matrix between transmitter k and the receiver. It is typically assumed that the entries of H_k are independent with mean zero and variance $(1/N)$. At time m ($m = 1, \ldots, M$) transmitter k transmits vector $x_k^{(m)} \in \mathbb{C}^{n_k}$ to the receiver, impaired by additive noise $\sigma w_k^{(m)} \in \mathbb{C}^{n_k}$ ($\sigma > 0$). It is also typically assumed that the entries of $x_k^{(m)}$ and $w_k^{(m)}$ are independent over position and time, and standardized. At time m the receiver senses

$$y^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} H_k x_k^{(m)} + \sigma w^{(m)}.$$

Let Y $(N \times M)$, X_k $(n_k \times M)$, and W $(N \times M)$ denote the respective matrices resulting from placing each of $y^{(m)}$, $x_k^{(m)}$, and $w^{(m)}$ as columns of a matrix. With $H = [H_1, \ldots, H_K]$ and $X = [X_1^T, \ldots, X_K^T]^T$, Y can be written as

$$Y = HP^{1/2}X + \sigma W,$$

where, with $n = n_1 + \cdots + n_K$, $P^{1/2}$ is $n \times n$ diagonal with its first n_1 entries $P_1^{1/2}$, next n_2 entries $P_2^{1/2}$, and so on.

The goal is to estimate the powers P_k from the matrix Y. This has been achieved by extending the work of X. Mestre ([2]), who has determined an effective way of estimating the population eigenvalues from those of the sampled ones of the matrix B_n described in the proposal. This work in turn relied on references [1], [2], and [17] of the proposal. Notice Y is the first N rows of

$$\begin{pmatrix} HP^{1/2} & I_N \\ 0_1 & 0_2 \end{pmatrix} \begin{pmatrix} X \\ W \end{pmatrix},$$

(I_N $N \times N$ identity matrix, 0_1, $n \times n$, 0_2 $n \times N$ zero matrices), so that results in [1], [2], would apply, if not for the fact that the entries of X and W are typically not of the same distribution. In [3] the extension of [1] and [2] to non iid entries is made, and the following is proven:

Theorem. Let $B_N = (1/M)YY^*$. Fix K, Assume $M > N$, $n < N$, $M/N \to c$, each $N/n_k \to c_k$, as $N \to \infty$ and certain assumptions on the size of c, and the c_k's. Let λ_i denote the i-th smallest eigenvalue of B_N and $s = (\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N})^*$. Then with probability 1 $\hat{P}_k \to P_k$ as $N \to \infty$ where

$$\hat{P}_k = \frac{NM}{n_k(M-N)} \sum_{i \in \mathcal{N}_k} (\eta_i - \mu_i),$$

where $\mathcal{N}_k = \{N - \sum_{i=k}^K n_i + 1, \ldots, N - \sum_{i=k+1}^K n_i\}$, the η_i's are the ordered eigenvalues of $\mathrm{diag}(\lambda_1, \ldots, \lambda_N) - (1/N)ss^*$, and the μ_i's are the ordered eigenvalues of $\mathrm{diag}(\lambda_1, \ldots, \lambda_N) - (1/M)ss^*$.

The paper demonstrates, through simulation, the superiority of this scheme over another method of estimation, which uses moment methods.

Mutual information. This work concentrated on establishing a central limit theorem on a quantity fundamental to MIMO (multiple input-multiple output) wireless communications systems. Consider a communications system with n transmiting antennas sending information to N receiving antennas. At an instant of time data $x_n \in \mathbb{C}^n$ is transmitted across the system, resulting on the received antennas the vector $y_n \mathbb{C}^N$ given by

$$y_n = H_n x_n + z_n.$$

Here, H_n, $N \times n$, is the *fading channel* matrix, and $z_n \in \mathbb{C}^N$ is white noise on the receiving antennas, having variance ρ for each of its components. In a *Rician* channel, it is assumed that

$$H_n = (1/\sqrt{n}) R_n^{1/2} X_n T_n^{1/2} + A_n,$$

where X_n is $N \times n$ containing iid complex Gaussian entries, R_n, $N \times N$, T_n $n \times n$, and A_n, $N \times n$ are deterministic with R_n and T_n Hermitian nonnegative definite ($R_n^{1/2}$ and $T_n^{1/2}$ being their nonnegative definite square roots). R_n and T_n are, respectively the correlations between the receive antennas and the transmit antennas. The A_n matrix allows for "direct line of sight" between each pair of transmit and receive antennas. According to the classical work of Shannon, the *mutual information* of the system is given by

$$I = \sum_{j=1}^{N} \log(\rho^{-1}\lambda_j + 1) = \log\det(\rho^{-1}H_nH_n^* + I),$$

where $\lambda_1 \dots, \lambda_N$ are the eigenvalues of $H_nH_n^*$. I is the rate at which, for any small probability, p, a coding scheme can be found for which the data can be transmitted with decoding error equal to p. Due to the entries of X_n being complex Gaussian, when dealing with I it is sufficient to assume R_n and T_n are both diagonal. Consider

$$\mathcal{I}_n(\rho) = (1/N)\log\det(H_nH_n^* + \rho I).$$

This quantity is the mutual information per receiver antenna. It follows that

$$I = N\mathcal{I}_n(\rho) - N\log\rho.$$

Work on the closeness of $\mathcal{I}_n(\rho)$ to a bounded deterministic value, $V_n(\rho)$, determined by R_n, T_n, and A_n as n and N approach infinity, has been done previously (Hachem, et.al. *Ann. Appl. Probab.* **17**(3), 2007, 875-930). It is assumed that

$$(**) \qquad 0 < \liminf_n \frac{N}{n} \le \limsup \frac{N}{n} < \infty.$$

The fluctuations of $\mathcal{I}_n(\rho)$ are important to understand, for example, in determining the probability of an *outage*, which is the inability to transmit at a given rate. The limiting distributional behavior of $\mathcal{I}_n(\rho)$ has been achieved in [4]. In the paper it is shown, when the entries of X_n are iid standardized with finite 16-th moment, and when $(**)$ holds, there exists a Θ_n, defined by R_n, T_n, and A_n ((2.3) of the paper) for which

$$\frac{N}{\sqrt{\Theta_n}}(\mathcal{I}_n(\rho) - \mathsf{E}\mathcal{I}_n(\rho)) \xrightarrow{\mathcal{D}} N(0,1),$$

as $n \to \infty$. The convergence of the bias term $N(\mathsf{E}\mathcal{I}_n(\rho) - V_n(\rho))$ is also shown to converge under certain assumptions. In particular, when the entries of X_n are complex Gaussian, it is shown that

$$N(\mathsf{E}\mathcal{I}_n(\rho) - V_n(\rho)) = O(N^{-1}).$$

This central limit theorem is the first assuming a Rician system, where A_n is non-zero.

Robust M-estimators. Research was also conducted on bringing together work on large dimensional sample covariance matrices and that of robust estimation of population covariance matrices. A large body of work on the latter has been developed for the purpose of handling outliers and heavy-tailed distributions. Let $x \in \mathbb{C}^n$ be a mean zero random vector with covariance matrice $C_n = \mathsf{E}(xx^*)$. The goal is to acquire an estimate of C_n from

$N > n$ independent samples, $x_1, \ldots, x_N$ of x. A *robust M-estimator*, $\hat{C}_n$ of C_n is defined to be a solution to

$$(**) \qquad \hat{C}_n = \frac{1}{N}\sum_{i=1}^{N} u\left(\frac{1}{n}x_i^*\hat{C}_n^{-1}x_i\right)x_ix_i^*$$

where $u \geq 0$ is a suitably chosen function. This matrix, known to exist, has displayed important robust properties, and converges with probability 1 to C_n as $N \to \infty$. The principal investigator worked on showing properties of $\hat{C}_n$ when n and N both increase to infinity, but are on the same order of magnitude. This was achieved in two cases. The first is where $x = x_n = A_n y_n$, where A_n is $n \times m$ with$A_n A_n* = C_n$ and $y_n \in \mathbb{C}^m$ consists of independent standardized entries. It was done by comparing $\hat{C}_n$ to the sample covariance matrix

$$\hat{S}_n = \frac{1}{N}\sum_{i=1}^{N} x_ix_i^*.$$

The following results are containing in [5]. Assume $u : \mathbb{R}^+ \to \mathbb{R}^+$ is nonincreasing, continuous, with the function $\phi(s) = su(s)$ nondecreasing, bounded with $\sup_s \phi(s) > 1$. Assume the existence of positive η and α such that for all n $\max_j \mathsf{E}(|y_{nj}|^{8+\eta}) < \alpha$, and with $c_n = n/N$, $\bar{c}_n = m/n$ assume

$$0 < \liminf_n c_n \leq \limsup_n c_n < 1 \quad \text{and} \quad \limsup_n \bar{c}_n < \infty.$$

Finally, assume all eigenvalues of C_n are contained in a fixed bounded interal away from 0 for all n. Then

Theorem 1: (I) There exists a unique solution, $\hat{C}_n$ to $(**)$ for all large n a.s. which can be obtained by

$$\hat{C}_n = \lim_{t\to\infty} Z^{(t)},$$

where $Z^{(0)} = I$ and

$$Z^{(t+1)} = \frac{1}{N}\sum_{i=1}^{N} u\left(\frac{1}{n}x_i^*(Z^{(t)})^{-1}x_i\right)x_ix_i^*.$$

(II) With $\|\cdot\|$ denoting spectral norm on matrices

$$\|\phi^{-1}(1)\hat{C}_n - \hat{S}_n\| \xrightarrow{a.s.} 0.$$

Using known properties on $\hat{S}_n$ Theorem 1 was applied to the direction of arrivals problem in array signal processing. Assume K signals are impinging on a collection of n sensors with angles $\theta_t, \ldots, \theta_K$. At each instant of time, t, assume the vector of data, $x_t \in \mathbb{C}^n$ at the sensors is of the form

$$x_t = \sum_{k=1}^{K} \sqrt{p_k}s(\theta_k)z_{k,t} + \sigma w_t,$$

where $s(\theta) \in \mathbb{C}^n$ is the known nonrandom unit norm steering vector for signals impinging the sensors at angle θ, $z_{k,t} \in \mathbb{C}$ is the signal source which is standardized with finite $8+\eta$ absolute moment, iid across t and independent across k, $p_k > 0$ is the transmit power of source k, and $\sigma w_t \in \mathbb{C}^n$ is the noise received at time t, where w containing iid entries, standardized, with finite $8+\eta$ absolute moments. Then $x_t = A_n y_t$, with $A_N = [S(\Theta)P^{1/2} \;\; \sigma I]$, $S(\Theta) = [s(\theta_1), \ldots, s(\theta_K)]$, $P = \operatorname{diag}(p_1, \ldots, p_K)$, and $y_t = (z_{1,t}, \ldots, z_{K,t}, w_t^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{C}^{n+K}$. Assume the above conditions on n, N, and $M = N + K$. It is straightforward to verify that C_n has smallest eigenvalue σ^2 with multiplicity $n - K$. Then the following is also proven.

Theorem 2: Let $E \in \mathbb{C}^{n\times(n-K)}$ be a matrix with columns containing orhogonal eigenvectors of C_n corresponding to eigenvalue σ^2. Let $\hat{e}_k$ denote the eigenvector of $\hat{C}_n$ with eigenvalue $\hat{\lambda}_k$, $\lambda_1 \leq \cdots \leq \lambda_n$. Then, as $n \to \infty$

$$\gamma(\theta) - \hat{\gamma}(\theta) \xrightarrow{a.s.} 0,$$

where

$$\gamma(\theta) = s(\theta)^* E E^* s(\theta)$$

$$\hat{\gamma}(\theta) = \sum_{i=1}^{n} \beta_i s(\theta)^* \hat{e}_i \hat{e}_i^* s(\theta)$$

and

$$\beta_i = \begin{cases} 1 + \sum_{k=n-K+1}^{n} \left(\frac{\hat{\lambda}}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right), & i \leq n-K \\ -\sum_{k=1}^{n-K} \left(\frac{\hat{\lambda}}{\hat{\lambda}_i - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_i - \hat{\mu}_k} \right), & i > n-K \end{cases}$$

with $\hat{\mu}_1 \leq \cdots \leq \hat{\mu}_n$ the eigenvalues of $\operatorname{diag}(\hat{\lambda}) - \frac{1}{n}\sqrt{\hat{\lambda}}\sqrt{\hat{\lambda}}^{\mathrm{T}}$, $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)^{\mathrm{T}}$.

In the classic MUSIC (multiple signal classification) algorithm, if C_n were known, γ would be used to determine the angles θ_j, being the zeros of γ. With Theorem 2 a robust estimate of γ in the large dimensional setting can be made where the estimates of the angles are made as the values which minimize $\hat{\gamma}$.

In [6] the above results were extended to the x_i being elliptically distributed. Specifically, it is assumed that $x_i = \sqrt{\tau_i} A_n y_n$, where A_n is as above except $n \leq m$, the τ_i are real random with some additional assumptions, and the y_i are independent unitarily invariant zero-mean vectors with $\|y_i\| = m$, independent of the τ_i. Assume also that $(\sup_s \phi(s))c_n < 1$. Then Theorem 1 of the previous paper holds true in this case, where now (II) is now

$$\|\hat{C}_n - \hat{S}_n\| \xrightarrow{a.s.} 0,$$

where

$$\hat{S}_n = \frac{1}{N} \sum_{i=1}^{N} v_n(\tau_i \gamma_n) x_i x_i^*,$$

with γ_n is the unique positive solution to the equation

$$1 = \frac{1}{N} \sum_{i=1}^{N} \frac{\psi_n(\tau_i \gamma)}{1 + c_n \psi_n(\tau_i \tau)}$$

and $v_n(x) = u_n(g_n^{-1}(x))$, $\psi_n(x) = xv_n(x)$, $g_n(x) = x/(1 - c_n\phi(x))$.

CLT of linear spectral statistics of a spiked population model. Consider the sample covariance matrix with a spiked population model discussed in the proposal: $\mathbf{B}_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$ where $T_n^{1/2}$, $n \times n$, is the Hermitian nonnegative definite square root of the nonnegative definite matrix T_n, X_n is $n \times N$ consisting of i.i.d. standardized entries, $c_n = n/N \to c > 0$, as $n \to \infty$, and for a fix M the eigenvalues of T_n are 1 with multiplicity $n - M$, the remaining M eigenvalues being different than 1. The limiting behavior of the M "spiked" eigenvalues is given in (2) of the proposal. Let H_n denote the empirical distribution function of the eigenvalues of T_n, and F^{c_n,H_n} the nonrandom distribution function corresponding to the empirical distribution function of the eigenvalues of $\mathbf{B}_n$, as described in the proposal: F^{c_n,H_n} is defined through $\hat{F}^{c_n,H_n}(x) = 1 - c_n + c_nF^{c_n,H_n}$ $(x \geq 0)$, which corresponds to the empirical distribution of the eigenvalues of $(1/N)X_n^*T_nX_n$, the Stieltjes transform $m_{\hat{F}^{c_n,H_n}}$ having a unique inverse given by

$$z(m) == \frac{1}{m} + c_n \int \frac{t}{1 + tm} dH_n(t).$$

In [7] the distributional behavior of linear statistics of eigenvalues of $\mathbf{B}_n$ for arbitrary T_n is studied where H_n converges in distribution to H. Let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $\mathbf{B}_n$. When the second and fourth moments of X_{11} match that of a real or complex normal random variable, it is shown that for analytic function f

$$(***) \qquad X_n(f) = n\left(\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i) - \int f(x)dF^{c_n,H_n}\right)$$

converges in distribution to a Gaussian random variable with known mean and variance depending on c and H. The convergence will be the same whether T_n is the identity matrix or of spiked behavior (M eigenvalues different than 1), since the limiting H will not change. However, the difference between the two F^{c_n,H_n} will be of order $1/n$ and in $(***)$ there is an n multiplying the difference between the linear statistic and the centering term, rendering a significant difference between the spiked and unspiked models.

In [8] the difference between the two F^{c_n,H_n} is derived up to an $0(1/n^2)$ term. It is given as Theorem 1 in the paper. It is applied to determine the power of a test introduced in [9]. Under Gaussian assumptions and when $c < 1$ A corrected likelihood ratio statistic

$$\tilde{L}^* = \operatorname{tr} \mathbf{B}_n - \log|\mathbf{B}_n - n$$

($\|\cdot\|$ denoting determinant) is introduced to test the hypothesis

$$H_0: \quad T = I \quad vs. \quad H_1: \quad T \neq I.$$

They prove in the paper that under H_0

$$\tilde{L}^* - n\left(1 - \frac{c_n - 1}{c_n}\log(1 - c_n)\right) \xrightarrow{\mathcal{D}} N(m(g), v(g)),$$

where

$$m(g) = -\frac{\log(1-c)}{2} \quad \text{and} \quad v(g) = -2\log(1-c) - 2c.$$

This will yield the probability of a type I error.

Because of Theorem 1 in [8], the power of the test can be determined under the specific alternative hypothesis H_1^*: $\mathbf{B}_n$ has M eigenvalues different than 1, having distinct eigenvalues $a_1 \geq \cdots \geq a_k$ with respective multipliciites $n_1, \ldots, n_k$. It is shown in [8] that, under H_1^*, the centering term $\int(x - \log x - 1)dF^{c_n,H_n}$ is

$$1 + \frac{1}{n}\sum_{i=1}^{k} n_i(a_i - \log a_i) - \frac{M}{n} - \left(1 - \frac{1}{c_n}\right)\log(1-c_n) + 0\left(\frac{1}{n^2}\right).$$

Therefore, from $(****)$, under H_1^*

$$\tilde{L}^* - n\int(x - \log x - 1)dF^{c_n,H_n} \xrightarrow{\mathcal{D}} N(m(g), v(g)).$$

With Φ denoting the standard distribution function, using significance level α (typically .05) it follows that the asymptotic power function of the test is

$$1 - \Phi\left(\Phi^{-1}(1-\alpha) - \frac{\sum_{i=1}^{k} n_i(a_i - 1 - \log a_i)}{\sqrt{-2\log(1-c) - 2c}}\right).$$

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